Weil-Petersson Volumes of the Moduli Spaces of CY Manifolds

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Abstract

In this paper it is proved that the volumes of the moduli spaces of polarized Calabi-Yau manifolds with respect to Weil-Petersson metrics are rational numbers. Mumford introduce the notion of a good metric on vector bundle over a quasi-projective variety in [11]. He proved that the Chern forms of good metrics define classes of cohomology with integer coefficients on the compactified quasi-projective varieties by adding a divisor with normal crossings. Viehweg proved that the moduli space of CY manifolds is a quasi-projective variety. The proof that the volume of the moduli space of polarized CY manifolds are rational number is based on the facts that the L^2 norm on the dualizing line bundle over the moduli space of polarized CY manifolds is a good metric. The Weil-Petersson metric is minus the Chern form of the L^2 metric on the dualizing line bundle. This fact implies that the volumes of Weil-Petersson metric are rational numbers. Also we get that the Weil-Petersson metric is a good metric. Therefore all the Chern forms define integer classes of cohomologies.

Contents

1 Introduction

1.1 General Remarks

There are several metrics naturally defined on the moduli space of Riemann surfaces. One of them is the Weil-Petersson metric. The Weil-Petersson metric is defined because of the existence of a metric with a constant curvature on the Riemann surface. Its curvature properties were studied by Ahlfors, Bers, S. Wolpert and so on.

The generalization of the Weil-Petersson metric on the moduli space of higher dimensional projective varieties was first introduced by Y.-T. Siu. He gave explicit formulas for the curvature of Weil-Petersson metric. See [12]. The generalization is possible thanks to the solution of Calabi conjecture due to Yau. See [18]. For Calabi-Yau manifolds it was noticed in [14] and [13] that the Weil-Petersson metric can be defined and computed by using the cup product of $(n-1,n)$ forms.

Another metric naturally defined on the moduli space of polarized CY manifolds is the Hodge metric. The holomorphic sectional curvature of the Hodge metric is negative and bounded away from zero. The holomorphic curvature of the Weil-Petersson metric is not negative. Recently some important results about the relations between the Weil-Petersson metric and Hodge metric were obtained. See [3], [8], [9] and [10].

Ph. Candelas and G. Moore asked if the Weil-Petersson volumes are finite. For the importance and the physical interpretation of the finiteness of the Weil-Petersson volumes to string theory see [2], [4] and [16]. In this paper we will answer Candelas-Moore question. Moreover we will prove that the Weil-Petersson volumes are rational numbers. I was informed by Prof. Lu that he and Professor Sun also proved the rationality of the volumes. See [10].

In 1976 D. Mumford introduced the notion of good metrics on vector bundles on quasi-projective varieties in [11]. He proved that the Chern forms of good metrics define classes of cohomology with integer coefficients on the compactified

quasi-projective varieties by adding a divisor with normal crossings. Viehweg proved that the moduli space of prioritized CY manifolds is a quasi-projective variety. The idea of this paper is to apply the results of Mumford to the moduli space of CY manifolds. We proved that the $\mathbf{L}^2$ metric on the dualizing sheaf is good. It was proved in [14] that the Chern form of the $\mathbf{L}^2$ metric on the dualizing sheaf defines the Weil-Petersson metric on the moduli space. See also [13]. So if we prove that the $\mathbf{L}^2$ metric on the dualizing sheaf is good then it will imply that the Weil-Petersson volumes are rational numbers. We will explain what is the meaning of a metric on a line bundle is good one.

According to [17] the moduli space of polarized CY manifolds $\mathfrak{M}_L(\mathrm{M})$ is a quasi-projective variety. Let $\overline{\mathfrak{M}_L(\mathrm{M})}$ be some projective compactification of $\mathfrak{M}_L(\mathrm{M})$ such that

$$\mathfrak{D} = \overline{\mathfrak{M}_L(\mathrm{M})} - \mathfrak{M}_L(\mathrm{M})$$

is a divisor of normal crossings. The meaning that the metric h on a line bundle over $\mathfrak{M}_L(\mathrm{M})$ is good is the following; Let $\tau_\infty \in \mathfrak{D}$, let D^N be an open polydisk containing τ_∞, then h is a good metric on some line bundle $\mathcal{L}$ defined on $\mathfrak{M}_L(\mathrm{M})$ if the curvature form of the metric of the line bundle around open sets

$$D^N - D^N \cap \mathfrak{D} = (D^*)^k \times D^{N-k}$$

is bounded from above by the Poincare metric on $(D^*)^k$ plus the standard metric on D^{N-k}. This implies that if we integrate the maximal power of the curvature form over $\mathfrak{M}_L(\mathrm{M})$ we get a finite number. Moreover such curvature forms are forms with coefficients distribution in the sense of Schwarz and they define classes of cohomology of $H^2\left(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z}\right)$.

Our proof that the L^2 metric h on the relative dualizing sheaf is a good metric is based on the construction of a canonical family of holomorphic forms ω_τ on the Kuranishi space given in [14]. The canonical family of holomorphic forms defines a special holomorphic local coordinates in the Kuranishi space where the components of the Weil-Petersson metric are given by

$$g_{i,\overline{j}} = \delta_{i,\overline{j}} + \frac{1}{6} R_{i,\overline{j},k,\overline{l}} \tau^k \overline{\tau^l} + ...$$

Since

$$h(\omega_\tau) = \|\omega_\tau\|^2 = (-1)^{\frac{n(n-1)}{2}} \left(-\sqrt{-1}\right)^n \int\limits_{\mathrm{M}} \omega_\tau \wedge \overline{\omega_\tau}$$

then around a point

$$\tau_\infty \in \mathfrak{D} = \overline{\mathfrak{M}_L(\mathrm{M})} - \mathfrak{M}_L(\mathrm{M})$$

we can compute explicitly h. Let D^N be any polydisk in $\overline{\mathfrak{M}_L(\mathrm{M})}$ containing τ_∞. Let

$$D^N - D^N \cap \mathfrak{D} = (D^*)^k \times D^{N-k}.$$

Let $\pi : \mathrm{U}_1 \times ... \times \mathrm{U}_k \to (D^*)^k$ be the uniformization map. From the results in [14] we deduce the following explicit formula for the pull back of the L^2 metric

on the relative dualizing sheaf h on $\mathrm{U}_1 \times ... \times \mathrm{U}_k \times D^{N-k}$:

$$\|\omega_\tau\|^2 \,|_{(D^*)^k\times D^{N-k}} := h|_{(D^*)^k\times D^{N-k}} =$$

$$\|\omega_\tau\|^2 := h(\tau,\overline{\tau}) := \sum_{i=1}^{k}\left(1-|\tau^i|^2\right) + \sum_{j=k+1}^{N}\left(1-|t^j|^2\right) + \phi(\tau,\overline{\tau}) + \Psi(t,\overline{t}), \quad (1)$$

for $0 \leq |\tau^i| < 1$, $0 \leq \left|t^j\right| < 1$ where $\phi(\tau,\overline{\tau})$ and $\Psi(t,\overline{t})$ are bounded real analytic functions on the unit disk. The expression (1) shows that the L^2 metric h is a good metric. This implies that the volumes of Weil-Petersson metrics are finite and they are rational numbers. Moreover it implies that Weil-Petersson metric is a good metric. So the Chern forms of it define classes of cohomologies in $H^{2k}\left(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z}\right)$ according to [11].

1.2 Description of the Content of the Paper

Next we are going to describe the content of each of the **Sections** in this article.

In **Section 2** we review the basis results from [14] and in [13] about local deformation theory of CY manifolds. We also review the results of [7] about the global deformation Theory.

In **Section 3** we review Mumford Theory of good metrics with logarithmic growth on vector bundles over quasi-projective varieties developed in [11].

In **Section 4** we prove that the L^2 metric on the dualizing line bundle over the moduli space is a good metric in the sense of Mumford. This results implies that the Weil-Petersson volumes are rational numbers.

1.3 Acknowledgements

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2 Moduli of Polarized CY Manifolds

2.1 Local Moduli

Let M be an even dimensional C^∞ manifold. We will say that M has an almost complex structure if there exists a section $I \in C^\infty(\mathrm{M}, Hom(T^*, T^*)$ such that $I^2 = -id$. T is the tangent bundle and T^* is the cotangent bundle on M. This definition is equivalent to the following one: Let M be an even dimensional C^∞ manifold. Suppose that there exists a global splitting of the complexified cotangent bundle $T^* \otimes \mathbf{C} = \Omega^{1,0} \oplus \Omega^{0,1}$, where $\Omega^{0,1} = \overline{\Omega^{1,0}}$. Then we will say that M has an almost complex structure. We will say that an almost complex

structure is an integrable one, if for each point $x \in$M there exists an open set $U \subset$M such that we can find local coordinates $z^1, .., z^n$, such that $dz^1, .., dz^n$ are linearly independent in each point $m \in U$ and they generate $\Omega^{1,0}|_U$.

Definition 1 *Let M be a complex manifold. Let $\phi \in \Gamma(M, Hom(\Omega^{1,0}, \Omega^{0,1}))$, then we will call ϕ a Beltrami differential.*

Since $\Gamma(\mathrm{M}, Hom(\Omega^{1,0}, \Omega^{0,1})) \backsimeq \Gamma(\mathrm{M}, \Omega^{0,1} \otimes T^{1,0})$, we deduce that locally ϕ can be written as follows: $\phi|_U = \sum \phi^{\beta}_{\overline{\alpha}} \overline{dz}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}$. From now on we will denote by A_ϕ the following linear operator:

$$A_\phi = \begin{pmatrix} id & \phi(\tau) \\ \overline{\phi(\tau)} & id \end{pmatrix}.$$

We will consider only those Beltrami differentials ϕ such that $det(A_\phi) \neq 0$. The Beltrami differential ϕ defines an integrable complex structure on M if and only if the following equation holds:

$$\overline{\partial}\phi = \frac{1}{2}\left[\phi, \phi\right], \tag{2}$$

where

$$\left[\phi, \phi\right]|_U := \sum_{\nu=1}^{n} \sum_{1 \leqq \alpha < \beta \leqq n} \left(\sum_{\mu=1}^{n} \left(\phi^{\mu}_{\overline{\alpha}} \left(\partial_\mu \phi^{\nu}_{\overline{\beta}} \right) - \phi^{\mu}_{\overline{\beta}} \left(\partial_\mu \phi^{\nu}_{\overline{\alpha}} \right) \right) \right) \overline{dz}^{\alpha} \wedge \overline{dz}^{\beta} \otimes \frac{\partial}{dz^{\nu}} \tag{3}$$

(See [5].) Kuranishi proved the following Theorem:

Theorem 2 *Let $\{\phi_i\}$ be a basis of harmonic $(0,1)$ forms of $\mathbb{H}^1(M, T^{1,0})$ on a Hermitian manifold M. Let G be the Green operator and let $\phi(\tau^1, .., \tau^N)$ be defined as follows:*

$$\phi(\tau) = \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2} \overline{\partial}^* G[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)]. \tag{4}$$

There exists $\varepsilon > 0$ such that for $\tau = (\tau^1, ..., \tau^N)$ such that $|\tau_i| < \varepsilon$ the tensor $\phi(\tau^1, ..., \tau^N)$ is a global C^∞ section of the bundle $\Omega^{(0,1)} \otimes T^{1,0}$.(See [5].)

2.2 Affine Flat coordinates in the Kuranishi Space

Based on Theorem 2, the following Theorem is proved in [14]:

Theorem 3 *Let M be a CY manifold and let $\{\phi_i\}$ be a basis of harmonic $(0,1)$ forms with coefficients in $T^{1,0}$. Then the equation (2) has a solution in the form:*

$$\phi(\tau) = \sum_{i=1}^{N} \phi_i \tau^i + \sum_{|I_N| \geqq 2} \phi_{I_N} \tau^{I_N} =$$

$$\sum_{i=1}^{N}\phi_i\tau^i+\frac{1}{2}\overline{\partial}^*G[\phi(\tau^1,...,\tau^N),\phi(\tau^1,...,\tau^N)] \quad (5)$$

and $\overline{\partial}^*\phi(\tau^1,...,\tau^N)=0$, $\phi_{I_N}\lrcorner\omega_M=\partial\psi_{I_N}$ *where* $I_N=(i_1,...,i_N)$ *is a multi-index,*

$$\phi_{I_N}\in C^\infty(M,\Omega^{0,1}\otimes T^{1,0}),\tau^{I_N}=(\tau^1)^{i_1}...(\tau^N)^{i_N}$$

and there exists $\varepsilon>0$ *such that when* $|\tau^i|<\varepsilon$ $\phi(\tau)\in C^\infty(M,\Omega^{0,1}\otimes T^{1,0})$ *where* $i=1,...,N$.

Definition 4 *Theorem 3 implies that the Kuranishi space* $\mathcal{K}$ *is defined as follows: Let* $\varepsilon>0$ *be such that the Beltarmi differentials* $\phi(\tau)$ *defined by* (5) *satisfy Theorem 2, then*

$$\mathcal{K}:\{\tau=(\tau^1,...,\tau^N)||\tau^i|<\varepsilon\}.$$

Thus $\tau=(\tau^1,...,\tau^N)$ *such that* $|\tau^i|<\varepsilon$ *is a local coordinate system in* $\mathcal{K}$*. It will be called the flat coordinate system in* $\mathcal{K}$*.*

It is a standard fact from Kodaira-Spencer-Kuranishi deformation theory that for each $\tau=(\tau^1,...,\tau^N)\in\mathcal{K}$ as in Theorem 3 the Beltrami differential $\phi(\tau^1,...,\tau^N)$ defines a new integrable complex structure on M. This means that the points of $\mathcal{K}$, where

$$\mathcal{K}:\{\tau=(\tau^1,...,\tau^N)||\tau^i|<\varepsilon\}$$

defines a family of operators $\overline{\partial}_\tau$ on the C^∞ family $\mathcal{K}\times M\to M$ and $\overline{\partial}_\tau$ are integrable in the sense of Newlander-Nirenberg. Moreover it was proved by Kodaira, Spencer and Kuranishi that we get a complex analytic family of CY manifolds $\pi:\mathcal{X}\to\mathcal{K}$, where as C^∞ manifold $\mathcal{X}\simeq\mathcal{K}\times$M. The family

$$\pi:\mathcal{X}\to\mathcal{K} \quad (6)$$

is called the Kuranishi family. The operators $\overline{\partial}_\tau$ are defined as follows:

Definition 5 *Let* $\{\mathcal{U}_i\}$ *be an open covering of M, with local coordinate system* $\{z_i^k\}$ *where* $k=1,...,\dim_{\mathbb{C}}M=n$*. We know that the Beltrami differential is given by:*

$$\phi(\tau)=\sum_{j,k=1}^{n}(\phi(\tau^1,...,\tau^N))_{\overline{j}}^{k}\ d\overline{z}^j\otimes\frac{\partial}{\partial z^k}.$$

Then it defines the $\overline{\partial_\phi}$ *operator associated with the new complex structure as follows:*

$$\left(\overline{\partial_\phi}\right)_{\tau,\overline{j}}=\frac{\overline{\partial}}{\overline{\partial z^j}}-\sum_{k=1}^{n}(\phi(\tau^1,...,\tau^N))_{\overline{j}}^{k}\frac{\partial}{\partial z^k}. \quad (7)$$

In [14] the following Theorems were proved:

Theorem 6 *There exists a family of holomorphic forms ω_τ of the Kuranishi family (6) such that in the coordinates $(\tau^1, ..., \tau^N)$ we have*

$$\omega_\tau = \omega_0 - \sum_{i,j} (\omega_0 \lrcorner \phi_i)\, \tau^i + \sum_{i,j} \omega_0 \lrcorner (\phi_i \wedge \phi_k)\, \tau^i \tau^j + O(3). \tag{8}$$

Theorem 7 *There exists a family of holomorphic forms ω_τ of the Kuranishi family (6) such that in the coordinates $(\tau^1, ..., \tau^N)$ we have*

$$\langle [\omega_\tau], [\omega_\tau] \rangle = (-1)^{\frac{n(n-1)}{2}} \left(\sqrt{-1}\right)^n \int_M \omega_\tau \wedge \overline{\omega_\tau} =$$

$$1 - \sum_{i,j} \langle \omega_0 \lrcorner \phi_i, \omega_0 \lrcorner \phi_j \rangle\, \tau^i \overline{\tau^j} +$$

$$\sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle\, \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5) =$$

$$1 - \sum_{i,j} \tau^i \overline{\tau^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle\, \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5)$$

and

$$\langle [\omega_\tau], [\omega_\tau] \rangle \leq \langle [\omega_0], [\omega_0] \rangle. \tag{9}$$

2.3 Weil-Petersson Metric

It is a well known fact from Kodaira-Spencer-Kuranishi theory that the tangent space $T_{\tau,\mathcal{K}}$ at a point $\tau \in \mathcal{K}$ can be identified with the space of harmonic (0,1) forms with values in the holomorphic vector fields $\mathbb{H}^1(\mathrm{M}_\tau, T)$. We will view each element $\phi \in \mathbb{H}^1(\mathrm{M}_\tau, T)$ as a point wise linear map from $\Omega^{(1,0)}_{\mathrm{M}_\tau}$ to $\Omega^{(0,1)}_{\mathrm{M}_\tau}$. Given ϕ_1 and $\phi_2 \in \mathbb{H}^1(M_\tau, T)$, the trace of the map

$$\phi_1 \circ \overline{\phi_2} : \Omega^{(0,1)}_{\mathrm{M}_\tau} \to \Omega^{(0,1)}_{\mathrm{M}_\tau}$$

at the point $m \in \mathrm{M}_\tau$ with respect to the metric g is simply given by:

$$Tr(\phi_1 \circ \overline{\phi_2})(m) = \sum_{k,l,m=1}^{n} (\phi_1)^k_{\overline{l}} (\overline{\phi_2)^m_{\overline{k}}} g^{\overline{l},k} g_{k,\overline{m}} \tag{10}$$

Definition 8 *We will define the Weil-Petersson metric on $\mathcal{K}$ via the scalar product:*

$$\langle \phi_1, \phi_2 \rangle = \int_M Tr(\phi_1 \circ \overline{\phi_2}) vol(g). \tag{11}$$

A very natural construction of a coordinate system $\tau = (\tau^1, ..., \tau^N)$ in $\mathcal{K}$ is constructed in [14] such that the components $g_{i,\overline{j}}$ of the Weil Petersson metric are given by the following formulas:

$$g_{i,\overline{j}} = \delta_{i,\overline{j}} + \frac{1}{6} R_{i,\overline{j},l,\overline{k}} \tau^l \overline{\tau^k} + O(\tau^3).$$

[1]

Very detailed treatment of the Weil-Petersson geometry of the moduli space of polarized CY manifolds can be found in [8] and [9]. In those two papers important results are obtained.

2.4 Global Moduli

Definition 9 *We will define the Teichmüller space $\mathcal{T}(M)$ of a CY manifold M as follows:* $\mathcal{T}(M) := \mathcal{I}(M)/Diff_0(M)$, *where*

$$\mathcal{I}(M) := \{\textit{all integrable complex structures on } M\}$$

and $Diff_0(M)$ is the group of diffeomorphisms isotopic to identity. The action of the group $Diff_0(M)$ is defined as follows; Let $\phi \in Diff_0(M)$ then ϕ acts on integrable complex structures on M by pull back, i.e. if $I \in C^\infty(M, Hom(T(M), T(M))$, then we define $\phi(I_\tau) = \phi^*(I_\tau)$.

We will call a pair $(M; \gamma_1, ..., \gamma_{b_n})$ a marked CY manifold where M is a CY manifold and $\{\gamma_1, ..., \gamma_{b_n}\}$ is a basis of H_n(M,$\mathbb{Z}$)/Tor.

Remark 10 *Let $\mathcal{K}$ be the Kuranishi space. It is easy to see that if we choose a basis of $H_n(M,\mathbb{Z})/Tor$ in one of the fibres of the Kuranishi family $\pi : \mathcal{X}_{\mathcal{K}} \to \mathcal{K}$ then all the fibres will be marked, since as a C^∞ manifold $\mathcal{X}_{\mathcal{K}} \cong M \times \mathcal{K}$.*

In [7] the following Theorem was proved:

Theorem 11 *There exists a family of marked polarized CY manifolds*

$$\mathcal{Z}_L \to \mathfrak{T}(M), \tag{12}$$

which possesses the following properties: ***a)*** *It is effectively parametrized,* ***b)*** *For any marked CY manifold M of fixed topological type for which the polarization class L defines an imbedding into a projective space $\mathbb{CP}^N$, there exists an isomorphism of it (as a marked CY manifold) with a fibre M_s of the family $\mathcal{Z}_L$.* ***c)*** *The base has dimension $h^{n-1,1}$.*

Corollary 12 *Let $\mathcal{Y} \to \mathfrak{X}$ be any family of marked polarized CY manifolds, then there exists a unique holomorphic map $\phi : \mathfrak{X} \to \mathfrak{T}(M)$ up to a biholomorphic map ψ of M which induces the identity map on $H_n(M, \mathbb{Z})$.*

From now on we will denote by $\mathcal{T}$(M) the irreducible component of the Teichmüller space that contains our fixed CY manifold M.

[1] This coordinate system is called flat holomorphic coordinate system. It appeared for the first time in [14]. Based on the information of the author of [13], it is claimed in [1] that the flat coordinate system was introduced in [13]. The problem of the construction of the flat holomorphic coordinates was not addressed in [13].

Definition 13 *We will define the mapping class group $\Gamma_1(M)$ of any compact C^∞ manifold M as follows:*

$$\Gamma_1(M) = Diff_+(M)/Diff_0(M),$$

where $Diff_+(M)$ is the group of diffeomorphisms of M preserving the orientation of M and $Diff_0(M)$ is the group of diffeomorphisms isotopic to identity.

Definition 14 *Let $L \in H^2(M,\mathbb{Z})$ be the imaginary part of a Kähler metric. We will denote by*

$$\Gamma_2 := \{\phi \in \Gamma_1(M) | \phi(L) = L\}.$$

It is a well know fact that the moduli space of polarized algebraic manifolds $\mathcal{M}_L(\mathrm{M}) = \mathcal{T}(\mathrm{M})/\Gamma_2$. In [7] the following fact was established:

Theorem 15 *There exists a subgroup of finite index Γ_L of Γ_2 such that Γ_L acts freely on $\mathcal{T}(M)$ and $\Gamma\backslash\mathcal{T}(M) = \mathfrak{M}_L(M)$ is a non-singular quasi-projective variety. Over $\mathfrak{M}_L(M)$ there exists a family of polarized CY manifolds $\pi : \mathcal{M} \to \mathfrak{M}_L(M)$.*

Remark 16 *Theorem 15 implies that we constructed a family of non-singular CY manifolds*

$$\pi : \mathcal{X} \to \mathfrak{M}_L(M) \tag{13}$$

over a quasi-projective non-singular variety $\mathfrak{M}_L(M)$. Moreover it is easy to see that $\mathcal{X} \subset \mathbb{CP}^N \times \mathfrak{M}_L(M)$. So $\mathcal{X}$ is also quasi-projective. From now on we will work only with this family.

Remark 17 *Theorem 15 implies that $\mathfrak{M}_L(M)$ is a quasi-projective non-singular variety. Using Hironaka's resolution theorem, we can find a compactification $\overline{\mathfrak{M}_L(M)}$ of $\mathfrak{M}_L(M)$ such that $\overline{\mathfrak{M}_L(M)} - \mathfrak{M}_L(M) = \mathfrak{D}$ is a divisor with normal crossings. We will call $\mathfrak{D}$ the discriminant divisor.*

2.5 Affine Flat Coordinates around Points at Infinity

Theorem 18 *Let $U_\infty = D^N \subset \overline{\mathfrak{M}_L(M)}$ be some open polydisk containing the point $\tau_\infty \in \mathfrak{D}$. Suppose that*

$$U_\infty - (U_\infty \cap \mathfrak{D}) = (D^*)^k \times D^{N-k}.$$

According to the results proved in [7] there exists a complete family of polarized CY manifolds

$$\pi : \mathcal{X} \to U_\infty - (U_\infty \cap \mathfrak{D}) = (D^*)^k \times D^{N-k}. \tag{14}$$

Let $\omega_{\mathcal{X}/U_\infty}$ be the relative dualizing line bundle. Then there exists a coordinate system $(\tau^1, ..., \tau^k, t^1, ..., t^{N-k})$ on the universal cover $(\mathrm{U})^k \times D^{N-k}$ of $(D^*)^k \times D^{N-k}$ and a global section $\omega_\tau \in \Gamma\left((\mathrm{U})^k, \pi_*\omega_{\mathcal{X}/(\mathrm{U})^k}\right)$ such that:

$$\omega_\tau = \omega_0 + \sum_{i=1}^{k} \omega_{i,0}(n-1,1)\tau^i + \sum_{i \le j=1}^{k} \omega_{ij,0}(n-2,2)\tau^i\tau^j + O(3) +$$

$$\sum_{j=1}^{n\_k}\omega_{j,0}(n-1,1)t^j + \sum_{i\leq j=1}^{N-k}\omega_{ij,0}(n-2,2)t^it^j + O(3). \tag{15}$$

Proof: The proof of Theorem 18 is based on the results obtained in [6].

Lemma 19 *Suppose that $\tau_\infty = 0 \subset U_\infty$ and D is an open disk in U_∞ containing 0. Suppose that the monodromy operator T of the restriction of the family (14) on $D - D\cap\mathfrak{D}$ is of infinite order. Then there exists a non zero section*

$$\omega_\tau \in \Gamma\left(U_\infty - (U_\infty \cap \mathfrak{D}), \pi_*\omega_{\mathcal{X}/U_\infty-(U_\infty\cap\mathfrak{D})}\right),$$

such that

$$\int_{\gamma_0}\omega_\tau = 1, \tag{16}$$

where γ_0 a a primitive invariant vanishing cycle with respect to the monodromy operator T.

Proof: Let us consider the family (14). Since we assumed that $D\cap U_\infty$ is any open disk and that U_∞ is a polydisk, then we can construct a non zero family of holomorphic forms Ω_t over $D\cap U_\infty$ according to [6]. So we can analytically extend this family to a family of holomorphic forms Ω_τ over U_∞. Thus we get:

$$\Omega_\tau \in \Gamma\left(\pi^{-1}\left(U_\infty - D\cap U_\infty\right), \omega_{\mathcal{X}/U_\infty}\right)$$

such that at each $\tau \in U_\infty - D\cap U_\infty$, $\Omega_\tau \neq 0$. According to Theorem **37** proved in [6] we have

$$\int_{\gamma_0}\Omega_t \neq 0 \text{ and } \lim_{t\to 0}\int_{\gamma_0}\Omega_t \neq 0 \tag{17}$$

for $t \in D$. (17) implies that the function $\phi(t) = \int_{\gamma_0}\Omega_\tau$ is different from zero on $U_\infty - D\cap U_\infty$. Then we can define $\omega_\tau = \frac{\Omega_\tau}{\phi(\tau)}$. Clearly the family of holomorphic n-forms ω_τ satisfies (16). Lemma 19 is proved. ■

Lemma 20 *Suppose that the monodromy operator T of the restriction of the family (14) on $D - D\cap\mathfrak{D} = \tau_\infty$ is of finite order m. Then there exists a $n-$cycle γ_0 and a non zero section $\omega_\tau \in \Gamma\left(U_\infty, \pi_*\omega_{\mathcal{X}/U_\infty}\langle\log\mathfrak{D}\rangle\right)$, such that on U_∞ we have*

$$\lim_{\tau\to 0}\int_{\gamma_0}\omega_\tau = 1. \tag{18}$$

Proof: Let $\phi_m : D \to D$ be the map $t \to t^m$. Let us pullback the restriction of the family (14) by ϕ_m. Then the monodromy operator T of the new family will be the identity. Then we can choose a $n-$cycle γ_0 such that

$$\lim_{t\to 0}\int_{\gamma_0}\Omega_t \neq 0.$$

The family of holomorphic forms Ω_t can be prolong to a family Ω_τ over $U_\infty - (U_\infty \cap \mathfrak{D})$ such that $\phi(\tau) := \int\limits_{\gamma_0} \Omega_\tau$ is a non zero function on U_∞. Then $\omega_\tau := \frac{\Omega_\tau}{\phi(\tau)}$ satisfies

$$\lim_{\tau \to 0} \int\limits_{\gamma_0} \omega_\tau = 1$$

Lemma 20 is proved. ■

We define the flat affine coordinates

$$\left(\tau^1, ..., \tau^k, t^1, ..., t^{N-k}\right)$$

in $(\mathrm{U})^k \times D^{N-k}$ as follows: Let ω_τ be the family of holomorphic $n-$ forms defined on U_∞ by Lemmas 19 and 20. Local Torelli Theorem implies that we can choose a basis of cycles

$$(\gamma_0, \gamma_1, ..., \gamma_N, \gamma_{N+1}, ..., \gamma_{2N+1}, ...\gamma_{b_n})$$

of $H_n\,(\mathrm{M},\mathbb{Z})$ satisfying

$$\langle \gamma_i, \gamma_j \rangle_{\mathrm{M}} = 0, \;\; \langle \gamma_i, \gamma_{2N+1-j} \rangle_{\mathrm{M}} = \delta_{ij}$$

for $i = 0, ..., k$; $j = 1, ..., N-k$ such that if

$$\tau^i := \int\limits_{\gamma_i} \omega_\tau, \;\; i = 1, ..., k \text{ and } t^j := \int\limits_{\gamma_{j+k}} \omega_\tau, \;\; j = 1, ..., N-k \tag{19}$$

then $(\tau^1, ..., \tau^k, t^1, ..., t^{N-k})$ will be a local coordinate system in $(\mathrm{U})^k \times (D)^{N-k}$.

Lemma 21 *Let $0 \in (\mathrm{U})^k$ be any fixed point. Then the Taylor expansion of the family of holomorphic n forms ω_τ constructed in Lemmas 19 and 20 satisfies (15).*

Proof: We know from [14] that we can identify the tangent space at $0 \in U_\infty - (U_\infty \cap \mathfrak{D})$ with $H^1(\mathrm{M}_0, T^{1,0}_{\mathrm{M}_0})$. The contraction with ω_0 defines an isomorphism

$$H^1\left(\mathrm{M}_0, T^{1,0}_{\mathrm{M}_0}\right) \approxeq H^1\left(\mathrm{M}_0, \Omega^{n-1,0}_{\mathrm{M}_0}\right).$$

Thus the tangent vectors

$$\phi_i = \frac{\partial}{\partial \tau^i} \in T_{0,U_\infty} = H^1\left(\mathrm{M}_0, \Omega^{n-1,0}_{\mathrm{M}_0}\right)$$

can be identified with classes of cohomologies $\omega_{i,0}(n-1,1) := \omega_0 \lrcorner \phi_i$ of type $(n-1,1)$. Griffiths' transversality implies that for $t=0$ we have

$$\left(\frac{\partial}{\partial \tau^i}\omega_\tau\right)|_{\tau=0} = a_0\omega_0 + \omega_0 \lrcorner \phi_i = a_0\omega_0 + \omega_{i,0}(n-1,1) \tag{20}$$

and

$$\left(\frac{\partial^2}{\partial \tau^i \partial \tau^j} \omega_\tau \right) |_{\tau=0} =$$

$$a_{i,j}(0)\omega_0 + b_{i,j}(0) \left(\omega_{i,j,0}(n-1,1) \right) + c_{ij}(0)\omega_{i,j,0}(n-2,2). \tag{21}$$

Proposition 22 *We have $a_0 = a_{i,j}(0) = b_{i,j}(0) = 0$ and $c_{ij}(0) = const \neq 0$ in the expression (21).*

Proof: The definition of the coordinates $(\tau^1, ..., \tau^k)$ and (20) and (18) imply that

$$a_0 = 0. \tag{22}$$

From (21) and (22) we can conclude that for $1 \leq i \leq k$ and $t = 0$ we have

$$\omega_\tau \Big|_{(\mathrm{U})^k} = \omega_0 + \sum_{i=1}^{k} \tau^i \left(\omega_0 \lrcorner \phi_i \right) +$$

$$\frac{1}{2} \sum_{i,j=1}^{k} \left(b_{ij}(0)\omega_{i,j,0}(n-1,1) + c_{ij}(0) \left(\omega_0 \lrcorner \phi_i \lrcorner \phi_j \right) \right) \tau^i \tau^j + ..., \tag{23}$$

where $b_{ij}(0)$ and $c_{ij}(0)$ are constants. So (18) implies

$$\int_{\gamma_0} \frac{\partial}{\partial \tau^i} \omega_\tau = \int_{\gamma_0} \frac{\partial^2}{\partial \tau^i \partial \tau^j} \omega_\tau = 0.$$

Since

$$\left(\frac{\partial}{\partial \tau^i} \omega_\tau \right) |_{\tau=0} = \omega_0 \lrcorner \phi_i := \omega_{i,0}(n-1,1),$$

and

$$\omega_\tau \Big|_{(\mathrm{U})^k} = \omega_0 + \sum_{i=1}^{k} a_i \tau^i \left(\omega_{i,0}(n-1,1) \right) + ...$$

we deduce that

$$\int_{\gamma_0} [\omega_{i,0}(n-1,1)] = 0 \text{ and } \int_{\gamma_j} [\omega_{i,0}(n-1,1)] = \delta_{ij}. \tag{24}$$

Thus (24) and (23) imply that $a_i = 1$.

The relations (19) and (23) imply that

$$\int_{\gamma_k} \omega_{i,j,0}(n-1,1) = 0 \tag{25}$$

for any γ_k such that $\int\limits_{\gamma_k} \omega_\tau = \tau^k$.Indeed (24) implies that for any non zero closed form $\omega(n-1,1)$ of type $(n-1,1)$ there exists γ_k such that

$$\int\limits_{\gamma_k} \omega(n-1,1) \neq 0. \tag{26}$$

Thus (25) and (26) imply that $b_{i,j}(0) = 0$. Proposition 22 is proved. ■

Proposition 22 implies Lemma 21. ■ Lemma 21 implies Theorem 18. ■

Remark 23 *Theorem 18 states that the coordinates used in the special geometry and the flat affine coordinates introduced in [14] by Theorems 3 and 7 are the same. This fact is mentioned in* **#5.1** *of [1]. In the same paper the authors referred to [30] (private communication by Tian) for the introduction of the flat affine coordinates.*

3 Metrics on Vector Bundles with Logarithmic Growth

3.1 Mumford Theory

In this **Section** we are going to recall some definitions and results from [11]. Let X be a quasi-projective variety. Let $\overline{\mathrm{X}}$ be a projective compactification of X such that $\overline{\mathrm{X}}-\mathrm{X}= \mathfrak{D}_\infty$ is a divisor with normal crossings. The existence of such compactification follows from the Hironaka's results. We will look at polydisk $D^N \subset \overline{\mathrm{X}}$, where D is the unit disk, $N = \dim \overline{\mathrm{X}}$ such that

$$D^N \cap \mathrm{X} = (D^*)^k \times \mathrm{D}^{N-k},$$

where $D^* = D - 0$ and q is the coordinate in D. On D^* we have the Poincare metric

$$ds^2 = \frac{|dq|^2}{|q|^2 (\log |q|)^2}.$$

On the unit disk D we have the simple metric $|dt|^2$. The product metric on $(D^*)^k \times \mathrm{D}^{N-k}$ we will call $\omega^{(P)}$.

A complex-valued C^∞ p-form η on X is said to have Poincare growth on $\overline{\mathrm{X}}-\mathrm{X}$ if there is a set of if for a covering $\{\mathcal{U}_\alpha\}$ by polydisks of $\overline{\mathrm{X}}-\mathrm{X}$ such that in each $\mathcal{U}_\alpha$ the following estimate holds:

$$\left|\eta\left(q^1,...,q^k,t^{k+1},...,t^N\right)\right| \leq C_\alpha \left\|\omega^{(p)}_{\mathcal{U}_\alpha}(q^1,\overline{q^1})\right\|^2 ... \left\|\omega^{(p)}_{\mathcal{U}_\alpha}(q^k,\overline{q^k})\right\|^2 \tag{27}$$

where

$$\left\|\omega^{(p)}_{\mathcal{U}_\alpha}(q^i,\overline{q^i})\right\|^2 = \frac{|q^i|^2}{|q^i|^2 (\log |q^i|)^2}$$

This property is independent of the covering $\{\mathcal{U}_\alpha\}$ of X but depends on the compactification $\overline{\mathrm{X}}$. If η_1 and η_2 both have Poincare growth on $\overline{\mathrm{X}}-\mathrm{X}$ then so does $\eta_1 \wedge \eta_2$. A complex valued C^∞ p-form η on $\overline{\mathrm{X}}$ will be called "good" on $\overline{\mathrm{X}}$ *if both* η and $d\eta$ have Poincare growth.

An important property of Poincare growth is the following:

Theorem 24 *Suppose that the η is a p-form with a Poincare growth on $\overline{X}-X=\mathfrak{D}_\infty$. Then for every C^∞ $(r-p)$ form ψ on $\overline{X}$ we have:*

$$\int_{\overline{X}} |\eta \wedge \psi| < \infty.$$

Hence, η defines a current $[\eta]$ on $\overline{X}$.

Proof: For the proof see [11]. ■

Definition 25 *Let $\mathcal{E}$ be a vector bundle on X with a Hermitian metric h. We will call h a good metric on $\overline{X}$ if the following holds.* ***1.*** *If for all $x\in \overline{X}-X$, there exist sections*

$$e_1, ..., e_m \in \mathcal{E}\left|_{D^N-(D^N\cap\mathfrak{D}_\infty)}\right.$$

of $\mathcal{E}$ which form a basis of $\mathcal{E}\left|_{D^N-(D^N\cap\mathfrak{D}_\infty)}\right.$. ***2.*** *In a neighborhood D^N of $x\in \overline{X}-X$ in which*

$$D^N \cap X = (D^*)^k \times D^{N-k}$$

and $\overline{X}-X=\mathfrak{D}_\infty$ is given by

$$t^1 \times ... \times t^N = 0$$

the metric $h_{i\overline{j}}=h(e_i,e_j)$ has the following properties: ***a.***

$$\left|h_{i\overline{j}}\right| \leq C\left(\sum_{i=1}^{k}\log\left|q^i\right|\right)^{2m}, \quad (\det(h))^{-1} \leq C\left(\sum_{i=1}^{k}\log\left|q^i\right|\right)^{2m} \tag{28}$$

for some $C>0$ and $m \geq 0$. ***b.*** *The 1-forms $\left((dh)\,h^{-1}\right)$ are good forms on $\overline{X}\cap D^N$.*

It is easy to prove that there exists a unique extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ on $\overline{\mathrm{X}}$, i.e. $\overline{\mathcal{E}}$ is defined locally as holomorphic sections of $\mathcal{E}$ which have a finite norm in h.

Theorem 26 *Let $(\mathcal{E},h)$ be a vector bundle with a good metric on $\overline{X}$, then the Chern classes $c_k(\mathcal{E},h)$ are good forms on $\overline{X}$ and the currents $[c_k(\mathcal{E},h)]$ represent the cohomology classes $c_k(\mathcal{E},h) \in H^{2k}\left(\overline{X},\mathbb{Z}\right)$.*

Proof: For the proof see [11]. ■

3.2 Example of a Good Metric

Theorem 27 *Let* $\pi : \mathrm{U}_1 \times ... \times \mathrm{U}_k \to (D^*)^k$ *be the uniformization map, where* U *is the unit disk. Suppose that*

$$\kappa_\infty \in (\partial(\mathrm{U}))^k = \left(S^1\right)^k,$$

Let h *be a metric on the line bundle* $\mathcal{L} \to (D^*)^k$. *Let*

$$\{\tau_m\} \in \mathrm{U}_1 \times ... \times \mathrm{U}_k$$

be any sequence such that

$$\lim_{m\to\infty} \tau_m = \kappa_\infty \in \overline{\mathrm{U}_1} \times ... \times \overline{\mathrm{U}_k} - \mathrm{U}_1 \times ... \times \mathrm{U}_k$$

and

$$\lim_{m\to\infty} \pi(\tau_m) = 0 \in (D)^k = \overline{(D^*)^k}.$$

Suppose that $\pi^*(h) = h_{\mathrm{U}^k}$ *is defined on* $U_1 \times ... \times \mathrm{U}_k$ *as follows:*

$$h_{\mathrm{U}^k} := \sum_{i=1}^{k} \left(1 - |\tau^i|^2\right) + \phi(\tau, \overline{\tau}), \tag{29}$$

where $\phi(\tau, \overline{\tau})$ *is a bounded* C^∞ *function on* $(\mathrm{U})^k$ *and*

$$\lim_{m\to\infty} \phi(\tau_m, \overline{\tau_m}) = \lim_{m\to\infty} h_{\mathrm{U}^k}(\tau_m, \overline{\tau_m}) = 0.$$

Then h *is a good metric in the sense of Mumford on the line bundle* $\mathcal{L} \to (D^*)^k$.

Proof: We need to show that h satisfies the conditions (28) and that $\partial \log h_{\mathrm{U}^k}$ is a good form. The conditions (28) followed immediately from the expression (27) for the metric defined by h_{U^k}. We need to show that h satisfies (27), i.e. $\partial \log h$ is a good form.

Lemma 28 *The* $(1,0)$ *form* $\partial \log h$ *is a good form.*

Proof: The definition of a good form implies that $\partial \log h$ is a good form on $(D^*)^k$ if and only iff $\partial \log h_{(\mathrm{U})^k}$ satisfies on the universal cover $(\mathrm{U})^k$ of $(D^*)^k$ the following inequalities on each unit disk $\mathrm{U}_i \subset (\mathrm{U})^k$:

$$0 \leq \frac{\frac{\partial}{\partial \tau^i} h_{\mathrm{U}_i}}{h_{\mathrm{U}_i}} \frac{\overline{\frac{\partial}{\partial \tau^i} h_{\mathrm{U}_i}}}{h_{\mathrm{U}_i}} \leq c \frac{1}{\left(1 - |\tau^i|^2\right)^2} \tag{30}$$

and

$$0 \leq \overline{\frac{\partial}{\partial \tau^i}} \left(\frac{\frac{\partial}{\partial \tau^i} h_{\mathrm{U}_i}}{h_{\mathrm{U}_i}} \right) \leq c \sum_{i=1}^{k} \frac{1}{\left(1 - |\tau^i|^2\right)^2}, \tag{31}$$

where $c > 0$. This statement follows from the fact that the pullback of the metric with a constant curvature on $(D^*)^k$ is the Poincare metric on $(\mathrm{U})^k$, i.e.

$$\pi^*\left(\sum_{i=1}^{k}\frac{\left|dq^i\right|^2}{\left|q^i\right|^2(\log|q^i|)^2}\right)=\sum_{i=1}^{k}\frac{\left|d\tau^i\right|^2}{(1-|\tau^i|^2)^2}$$

and $\partial\log\pi^*(h)=\partial\log h_{(\mathrm{U})^k}$.

Proposition 29 *The form $\partial\log h_{(\mathrm{U})^k}$ satisfies (30) and (31).*

Proof: (30) and (31) will follow if we prove that the restriction of $\partial\log h_{(\mathrm{U})^k}$ on each U_i satisfies (30) and (31) . Direct computations show that the expression (29) of $h_{(\mathrm{U})^k}$ implies that we have :

$$h_i\left(\tau^i,\overline{\tau^i}\right):=h_{(\mathrm{U})^k}|_{\mathrm{U}_i}=\left(1-|\tau|^2\right)+\phi_i\left(\tau^i,\overline{\tau^i}\right)>0, \tag{32}$$

where

$$\lim_{\tau^i\to\kappa^i_\infty}\phi_i(\tau^i,\overline{\tau^i})=\lim_{\tau^i\to\kappa^i_\infty}h_i(\tau^i,\overline{\tau^i})=0$$

and $\phi_i(\tau^i,\overline{\tau^i})$ is a bounded C^∞ function on U_i. (32) implies:

$$0\le\left(1-|\tau^i|^2\right)\le C_ih_i\left(\tau^i,\overline{\tau^i}\right), \tag{33}$$

where $C_i > 0$. Thus we get from (33) that if τ_0 is any complex number such that $|\tau_0| = 1$ then the limit

$$\lim_{\tau^i\to\tau_0}\frac{(1-|\tau^i|^2)}{h_i\left(\tau^i,\overline{\tau^i}\right)}$$

exists and

$$0\le\lim_{\tau^i\to\tau^i_0}\frac{(1-|\tau^i|^2)}{h_i\left(\tau^i,\overline{\tau^i}\right)}=c. \tag{34}$$

Direct computations show that

$$\frac{\partial}{\partial\tau^i}\log h_{h_{\mathrm{U}_i}}=\frac{\frac{\partial}{\partial\tau^i}\left(1-|\tau^i|^2+\phi_i\left(\tau^i,\overline{\tau^i}\right)\right)}{\left(1-|\tau^i|^2+\phi_i\left(\tau^i,\overline{\tau^i}\right)\right)}=$$

$$\frac{\left(-\overline{\tau^i}+\frac{\partial}{\partial\tau^i}\phi_i\left(\tau^i,\overline{\tau^i}\right)\right)}{\left(1-|\tau^i|^2+\phi_i\left(\tau^i,\overline{\tau^i}\right)\right)}. \tag{35}$$

We derive from (34) , (35) and the fact that ϕ_i is bounded C^∞ function on U_i that we have:

$$0\le\frac{\frac{\partial}{\partial\tau^i}h_{h_{\mathrm{U}_i}}}{h_{\mathrm{U}_i}}\overline{\frac{\frac{\partial}{\partial\tau^i}h_{\mathrm{U}_i}}{h_{\mathrm{U}_i}}}_i\le c_1\frac{1}{(1-|\tau^i|^2)^2}$$

and

$$0 \leq \left| \overline{\frac{\partial}{\partial \tau^i}} \left(\frac{\partial h_{\mathrm{U}_i}}{h_{\mathrm{U}_i}} \right) \right| \leq c_1 \frac{1}{\left(1 - |\tau^i|^2\right)^2}, \tag{36}$$

where $c_1 > 0$. Thus (36) implies that $\partial \log h$ defines a good form on the line bundle $\mathcal{L}$ restricted on $(D^*)^k$. Lemma 28 is proved. ■

Lemma 28 implies Theorem 27. ■

4 Applications of Mumford Theory to the Moduli of CY

4.1 The L^2 Metric is Good

We are going to prove the following result:

Theorem 30 *The natural L^2 metric :*

$$h(\tau, \overline{\tau}) = \|\omega_\tau\|^2 := (-1)^{\frac{n(n-1)}{2}} \left(\sqrt{-1}\right)^n \int_M \omega_\tau \wedge \overline{\omega_\tau} \tag{37}$$

on $\pi_* \left(\omega_{\mathcal{X}(M)/\mathfrak{M}_L(M)}\right) \to \mathfrak{M}_L(M)$ is a good metric.

Outline of the proof of Theorem 30. Let $(D)^N$ be a polydisk in $\overline{\mathfrak{M}_L(\mathrm{M})}$ such that $0 \in (D)^N \cap \mathfrak{D} \neq \varnothing$, where $N = \dim_{\mathbb{C}} \mathfrak{M}_L(\mathrm{M})$. To prove Theorem 30 we need to derive an explicit formula for the metric $\langle \omega_\tau, \omega_\tau \rangle := h(\tau, \overline{\tau})$ on the line bundle $\pi_* \left(\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\right)$ restricted on

$$(D)^N - \left((D)^N \cap \mathfrak{D}\right) = (D^*)^k \times (D)^{N-k}.$$

Let $(\mathrm{U}^i)^k \times (D)^{N-k}$ be the universal cover of

$$(D)^N - \left((D)^N \cap \mathfrak{D}\right) = (D^*)^k \times (D)^{N-k},$$

where U^i are the unit disks. Let

$$\pi : (\mathrm{U}^i)^k \times (D)^{N-k} \to (D^*)^k \times (D)^{N-k} \tag{38}$$

be the covering map.

We will prove that formula (9) implies that we have the following expression for the L^2 metric $\langle \omega_\tau, \omega_\tau \rangle := h(\tau, \overline{\tau})$ on $\pi_* \left(\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\right)$ restricted on the universal covering of $(\mathrm{U}^i)^k \times (D)^{N-k}$ of $(D^*)^k \times (D)^{N-k}$:

$$\langle \omega_\tau, \omega_\tau \rangle := h(\tau, \overline{\tau}) :=$$

$$\sum_{i=1}^{k} \left(1 - |\tau^i|^2\right) + \sum_{j=k+1}^{N} \left(1 - |t^j|^2\right) + \phi(\tau, \overline{\tau}) + \Psi(t, \overline{t}) \tag{39}$$

where $\phi(\tau, \overline{\tau})$, $\Psi(t, \overline{t})$ are bounded real analytic functions on $(\mathrm{U}^i)^k \times (D)^{N-k}$. Theorem 27 and formula (39) imply Theorem 30.

Proof: The proof will follow from the Lemma proved bellow.

Lemma 31 *Let* $(\mathrm{U})^k$ *be the universal cover of* $(D^*)^k := (D)^k - (D)^k \cap \mathfrak{M}_L(M)$ *where* U *is the unit disk. Then*

i. *There exists a family of CY manifolds*

$$\pi^*\left(\mathcal{X}_{(\mathrm{U})^k}\right) \to (\mathrm{U})^k. \tag{40}$$

over $(\mathrm{U})^k$ *and a holomorphic section* $\omega \in H^0\left((\mathrm{U})^k, \pi_*\omega_{\mathcal{X}(\mathrm{M})/(\mathrm{U})^k}\right)$ *such that* $\omega_\tau = \omega|_{\mathrm{M}_\tau}$ *is a non zero holomorphic form on* M_τ.

ii. $\langle\omega_\tau, \omega_\tau\rangle$ *can be represented on* $(\mathrm{U})^k$ *as follows*:

$$\langle[\omega_\tau], [\omega_\tau]\rangle = h(\tau, \overline{\tau}) = \sum_{i=1}^{k}\left(1 - \left|\tau^i\right|^2\right) + \phi(\tau, \overline{\tau}), \tag{41}$$

where $\phi(\tau, \overline{\tau})$ *is a bounded real analytic functions on* $(\mathrm{U})^k$ *such that the limits*

$$\lim_{\tau \to \kappa_\infty \in \left(\overline{\mathrm{U}}\right)^k - (\mathrm{U})^k} h(\tau, \overline{\tau}) \text{ and } \lim_{\tau \to \kappa_\infty \in \left(\overline{\mathrm{U}}\right)^k - (\mathrm{U})^k} \phi(\tau, \overline{\tau})$$

exist where

$$\kappa_\infty = \left(\kappa_\infty^1, ..., \kappa_\infty^k\right) =$$

$$\lim_{(q^1, \ldots, q^k) \to (0, \ldots, 0)} \left(\pi^{-1}(q^1) = \tau^1, ..., \pi^{-1}(q^1) = \tau^k\right),$$

$$(q^1, ..., q^k) \in (D^*)^k, \; \pi(\kappa_\infty) = (0, ..., 0) \in \left(\overline{D^*}\right)^k = D^k$$

and $\left(\kappa_\infty^1, ..., \kappa_\infty^k\right) \in (\partial \mathrm{U})^k$.

iii. *Suppose that* $\lim_{\tau \to \kappa_\infty \in \left(\overline{\mathrm{U}}\right)^k - (\mathrm{U})^k} h(\tau, \overline{\tau}) = 0$. *Then*

$$\lim_{\tau \to \kappa_\infty \in \left(\overline{\mathrm{U}}\right)^k - (\mathrm{U})^k} \phi(\tau, \overline{\tau}) = 0.$$

iv. *The function* $\phi(\tau, \overline{\tau})$ *is bounded on* $\left(\overline{\mathrm{U}}\right)^k$.

Proof of i: Since $(D^*)^k \subset \mathfrak{M}_L(\mathrm{M})$, Remark 16 implies that there exists a family of CY manifolds

$$\pi : \mathcal{X}_{(D^*)^k} \to (D^*)^k \tag{42}$$

over $(D^*)^k$. Because $p : (\mathrm{U})^k \to (D^*)^k$ is the universal cover of $(D^*)^k$, then the pull back of the family (42) by p defines the family (40). Let ω_τ be the section of the pullback of the restriction of the relative dualizing line bundle $\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ on $(D^*)^k$ on $(\mathrm{U})^k$ constructed in Theorem 18. ■

Proof of ii: The proof of Part **ii** is based on the following Proposition:

Proposition 32 *Let us consider* $(D_{\alpha_1,\alpha_2})^k \subset (D^*)^k \subset (D)^k \subset \mathfrak{M}_L(M)$, *where*

$$D_{\alpha_1,\alpha_2} := \left\{t \in D_{\alpha_1,\alpha_2} \,||t| < 1 \text{ and } \alpha_1 < \arg t < \alpha_2\right\}.$$

Suppose the closure of $(D_{\alpha_1,\alpha_2})^k$ *in* $(D)^k$ *contains* $0 \in \left(\overline{D^*}\right)^k = D^k$. *Let us consider the restriction of the family (40)*

$$\mathcal{X}_{\alpha_1,\alpha_2} \to (D_{\alpha_1,\alpha_2})^k \tag{43}$$

on $(D_{\alpha_1,\alpha_2})^k \subset \mathfrak{M}_L(M)$. *Let* $\omega_{\mathcal{X}_{\alpha_1,\alpha_2}/(D_{\alpha_1,\alpha_2})^k}$ *be the restriction of dualizing sheaf of the family of polarized CY manifolds (40) on* $(D_{\alpha_1,\alpha_2})^k$. *Then there exists a global section*

$$\eta \in \Gamma\left((D_{\alpha_0,\alpha_1})^k, \pi_*\omega_{\mathcal{X}_{\alpha_1,\alpha_2}/(D_{\alpha_1,\alpha_2})^k}\right)$$

such that the classes of cohomology $[\eta_q]$ *defined by the restriction of* η *on all of the fibres* $\pi^{-1}(q) := M_q$ *for* $q \in (D_{\alpha_1,\alpha_2})^k$ *are non zero elements of* $H^0(M_q, \Omega^n_{M_q})$. *The limit* $\lim\limits_{q\to 0}[\eta_q]$ *exists and*

$$\lim_{q\to 0}[\eta_q] = [\eta_0] \text{ and } \langle [\eta_0], [\eta_0]\rangle \geq 0. \tag{44}$$

Proof: $(D_{\alpha_1,\alpha_2})^k$ is a contractible sector in $(D^*)^k$. Thus if we fix a basis $(\gamma_1, ..., \gamma_{b_n})$ in $H^n(\mathrm{M},\mathbb{Z})/Tor$ then we are fixing the marking of the family (43) over each point M_τ for each point $\tau \in (D_{\alpha_1,\alpha_2})^k$. This means that the basis $(\gamma_1, ..., \gamma_{b_n})$ of $H^n(\mathrm{M},\mathbb{Z})/Tor$ is defined and fixed on $H^n(\mathrm{M}_\tau,\mathbb{Z})/Tor$ on each fibre of the family (43). Now we can define the period map of the family by

$$p(q) := \left(..., \int_{\gamma_i} \eta_q, ...\right),$$

where η_q is a non zero holomorphic form on $\pi^{-1}(q) := \mathrm{M}_q$. Local Torelli Theorem implies that the period map p of marked CY manifolds

$$p : (D_{\alpha_1,\alpha_2})^k \to \mathbb{P}(H^n(\mathrm{M},\mathbb{C})). \tag{45}$$

is an embedding $(D_{\alpha_1,\alpha_2})^k \subset \mathbb{P}(H^n(\mathrm{M},\mathbb{C}))$. Thus we can conclude from the compactness of $\mathbb{P}(H^n(\mathrm{M},\mathbb{C}))$ and (45) the existence of a sequence of $[\eta_q]$ such that

$$\lim_{q\to 0}[\eta_q] = [\eta_0] \tag{46}$$

exists and $\eta|_{\mathrm{M}_0} \neq 0$. (46) implies (44). ■

Corollary 33 *There exists a global section* $\eta \in H^0\left((\mathrm{U})^k, \omega_{\mathcal{X}_{(\mathrm{U})^k}/(\mathrm{U})^k}\right)$ *such that* $\lim\limits_{\tau\to\kappa_\infty}[\eta_\tau]$ *exists and*

$$\lim_{\tau\to\kappa_\infty}[\eta_\tau] = [\eta_{\kappa_\infty}] \neq 0. \tag{47}$$

Proposition 34 *Let $\{\omega_\tau\}$ be the family of holomorphic $n-$forms constructed in Theorem 18 on the family restricted on $(\mathrm{U})^k$. Then the limit* $\lim\limits_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k}[\omega_\tau]$ *exists,*

$$\lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k} = [\omega_\infty] \ \ \textit{and} \ \ \langle[\omega_\infty],[\omega_\infty]\rangle \geq 0. \tag{48}$$

Proof: According to Corollary 33 there exists a global section

$$\eta \in H^0\left((\mathrm{U})^k, \omega_{\mathcal{X}_{(\mathrm{U})^k}/(\mathrm{U})^k}\right)$$

such that $\lim\limits_{\tau\to\kappa_\infty}[\eta_\tau]$ satisfies (47). The relation between the cohomologies of holomorphic forms $\eta_\tau := \eta_q$ and ω_τ are given by the formula $[\eta_\tau] = \varphi(\tau)[\omega_\tau]$, where $\varphi(\tau)$ is a holomorphic function on the product $(\mathrm{U})^k$. According to Theorem 7 we have

$$0 \leq \langle[\omega_\tau],[\omega_\tau]\rangle \leq \langle[\omega_{\tau_0}],[\omega_{\tau_0}]\rangle. \tag{49}$$

Thus (45), (49) and $[\eta_\tau] = \varphi(\tau)[\omega_\tau]$ imply formula (48). So the limit

$$\lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k}\langle[\omega_\tau],[\omega_\tau]\rangle$$

exists and

$$\lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k}\langle[\omega_\tau],[\omega_\tau]\rangle = \lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k} h(\tau,\overline{\tau}) = h(\kappa_\infty) \geq 0.$$

Proposition 34 is proved. ∎

Corollary 35 *Let $[\omega_\infty]$ be defined by (48). Then $\langle[\omega_\infty],[\omega_\infty]\rangle = 0$ if and only if the monodromy of the restriction of the family (13) is infinite.*

Notice that the functions $\langle\langle[\eta_\tau],[\eta_\tau]\rangle\rangle$ and $\langle[\omega_\tau],[\omega_\tau]\rangle$ are real analytic. If we normalize ω_0 and ϕ_i such that

$$\|\omega_{\tau_0}\|^2 = \langle\omega_0,\omega_0\rangle = 1$$

and

$$\langle\omega_0\lrcorner\phi_i,\omega_0\lrcorner\phi_j\rangle = \delta_{i\overline{j}}$$

we get from (9) the following expression

$$h(\tau,\overline{\tau}) =$$

$$1-\sum_{i=1}^k|\tau^i|^2 + \sum_{i\leq j}\left\langle\omega_0\lrcorner\left(\phi_i\wedge\phi_k\right),\omega_0\lrcorner\left(\phi_j\wedge\phi_l\right)\right\rangle\tau^i\overline{\tau^j}\tau^k\overline{\tau^l} + O(\tau^5) =$$

$$1-\sum_{i=1}^k|\tau^i|^2 + \Phi(\tau,\overline{\tau}) \tag{50}$$

holds. Also (50) implies that the restriction of $\langle[\omega_\tau],[\omega_\tau]\rangle = h(\tau,\overline{\tau})$ on the universal cover $(\mathrm{U})^k$of $(D^*)^k$ will be given by (41), i.e.

$$\langle[\omega_\tau],[\omega_\tau]\rangle = h(\tau,\overline{\tau}) = 1-\sum_{i=1}^{k}\left|\tau^i\right|^2+\Phi(\tau,\overline{\tau}).$$

Proof of (41) : We can rewrite the above expression as follows:

$$\langle[\omega_\tau],[\omega_\tau]\rangle = h(\tau,\overline{\tau}) = 1-\sum_{i=1}^{k}\left|\tau^i\right|^2+\Phi(\tau,\overline{\tau}) =$$

$$\sum_{i=1}^{k}\left(1-\left|\tau^i\right|^2\right)-k+1+\Phi(\tau,\overline{\tau}) = \sum_{i=1}^{k}\left(1-\left|\tau^i\right|^2\right)+\phi(\tau,\overline{\tau}). \tag{51}$$

where $\phi(\tau,\overline{\tau}) = \Phi(\tau,\overline{\tau})-k+1$. Proposition 34 and (48) imply that

$$\lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k} h(\tau,\overline{\tau}),\quad \lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k}\phi(\tau,\overline{\tau})$$

exist and $\phi(\tau,\overline{\tau})$ is a bounded real analytic function on $(\mathrm{U})^k$. Part **ii** of Lemma 31 is proved. ■

Proof of part iii: Suppose that U is the unit disk and

$$\lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k}\langle\omega_\tau,\omega_\tau\rangle = \lim_{\tau\to\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k} h(\tau,\overline{\tau}) = 0.$$

Notice that since $\kappa_\infty\in\left(\overline{\mathrm{U}}\right)^k-(\mathrm{U})^k$, where U is the unit disk then for each i we have $\left(1-\left|\kappa_\infty^i\right|^2\right)=0$ and thus

$$\sum_{i=1}^{k}\left(1-\left|\kappa_\infty^i\right|^2\right)=0. \tag{52}$$

Thus (51) and (52) imply (41) . Part **iii** is proved. ■

Part **iii** implies Part **iv.** ■

Lemma 31 is proved. ■

Lemma 36 *Suppose that the L^2 metric on the relative dualizing sheaf defined by the function $h(\tau,\overline{\tau})=\langle\omega_\tau,\omega_\tau\rangle$ on*

$$D^N-\left(D^N\cap\mathfrak{D}\right)=(D^*)^k\times D^{N-k}\subset\mathfrak{M}_L(M).$$

is bounded on D^N, $h|_{(D)^N-(D)^N\cap\mathfrak{D}}>0$ and $h|_{(D)^N\cap\mathfrak{D}}\geq 0$. Then the L^2 metric is good.

Proof: The proof of Lemma 36 is obvious. ■

The expression (41) for the L^2 metric and Theorem 27 implies that if

$$h|_{(D)^N\cap\mathfrak{D}}\geq 0$$

then the L^2 metric is a good metric . Theorem 30 is proved. ■

4.2 The Weil-Petersson Volumes are Rational Numbers

Theorem 37 *The Weil-Petersson volume of the moduli space of polarized CY manifolds is finite and it is a rational number.*

Proof: Theorem 30 implies that the metric on the relative dualizing sheaf $\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ defined by (37) is a good metric. This implies that the Chern form of any good metric defines a class of cohomology in

$$H^2\left(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z}\right) \cap H^{1,1}\left(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z}\right).$$

See Theorem 26. We know from [14] that the Chern form of the metric h is equal to minus the imaginary part of the Weil-Petersson metric. So the imaginary part of the Weil-Petersson metric is a good form in the sense of Mumford. This implies that

$$\int\limits_{\overline{\mathfrak{M}_L(\mathrm{M})}} \wedge^{\dim_{\mathbb{C}} \overline{\mathfrak{M}_L(\mathrm{M})}} c_1(h) \in \mathbb{Z}$$

since $\overline{\mathfrak{M}_L(\mathrm{M})}$ is a smooth manifold. Since $\mathfrak{M}_L(\mathrm{M})$ is a finite cover of the moduli space $\mathcal{M}_L(\mathrm{M})$ then the Weil-Petersson volume of $\mathcal{M}_L(\mathrm{M})$ will be a rational number. Theorem 37 is proved. ■

In the paper [10] the authors proved that the Weil-Petersson volumes of the moduli space of CY manifolds are finite.

Corollary 38 *The Weil-Petersson metric is a good metric on the moduli space* $\overline{\mathfrak{M}_L(M)}$ *and the Chern forms* $c_k\,[W.-P.]$ *of the Weil-Petersson metric are well defined elements of* $H^{2k}\left(\overline{\mathfrak{M}_L(M)}, \mathbb{Z}\right)$.

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